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The Discrete Evolution Model of Bak and Sneppen is Conjugate to the Classical Contact Process

Christoph Bandt[1]

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Two fundamental models of critical phenomena are connected. We show that the discrete Bak–Sneppen evolution model is conjugate to the classical contact process. This holds in discrete and continuous time, on all graphs and for random as well as for deterministic choice of neighbors. Thus the extensive theory for the contact process applies to the discrete Bak–Sneppen model, too.

KEY WORDS: Contact process; cellular automata; thinning; self-organized criticality; evolution model.

The contact process, CP, was worked out by Harris[7] in 1972. It models the spatial-temporal development of an infectious disease, a rumor or a new trend of fashion. Individuals are represented by the vertices of a graph, and edges connect "neighbors" which have contact with each other. In the standard one-dimensional case, vertices are the integers and edges lead from n to $n-1$ and $n+1$, for all n. Each individual can be in one of two states 0 and 1, healthy and sick. Sick individuals will recover with rate 1, and will infect a randomly chosen neighbor with rate $\lambda > 0$. An extensive mathematical theory, collected in the monographs of Liggett,[9,10] allows to study this process in continuous time for infinite configurations of 1s. Among others, critical parameters have been estimated, it was shown that "the critical contact process dies out" and that on the homogeneous tree CP admits two phase transitions.[14]

On a finite graph, the process can be studied in discrete time as follows. At each step, choose randomly one of the individuals, with equal

[1]Institute of Mathematics, University of Greifswald, Germany; e-mail: bandt@uni-greifswald.de

probability. If it is healthy, nothing happens. If it is sick, we choose one of its neighbors, also with equal probability. Now with probability $q = 1/(1+\lambda)$ let the chosen individual recover, that is $1 \mapsto 0$, and with probability $p = \lambda/(1+\lambda)$ let the individual infect the chosen neighbor, provided this neighbor is healthy: $10 \mapsto 11$. For small λ, the disease will die out, so that only zeros remain. For large λ the disease will spread, so that when time tends to infinity, for each individual the probability to be sick is larger than some positive constant ε. In-between there is the critical parameter λ^c.

A model of evolution, BS, was introduced by Bak and Sneppen in 1993.[2] It represents species as vertices of a graph. Edges connect "neighboring species" which depend on each other, like prey and predator. Again, the standard case is that species are numbered by integers, and n is a neighbor of $n-1$ and $n+1$. The state of each species n is a number x_n between 0 and 1, the so-called fitness. At each discrete time step, the species n^* with minimum fitness is chosen. It will die out, and is immediately replaced by another species with random fitness. Thus x_{n^*} is replaced by a random number equidistributed in $[0, 1]$. At the same time, the neighboring species will also become extinct, even though they may have high fitness, because of their connection with the eliminated species. They are replaced by new species, too. Thus in the standard setting, x_{n^*-1}, x_{n^*} and x_{n^*+1} are replaced by equidistributed random numbers in $[0, 1]$. Moreover, all random numbers should be independent.

BS is a model of self-organized criticality which means that it will automatically approach its critical state. Computer simulations show that for a large number of species and sufficiently long time, the fitness values of most species become equidistributed above a certain critical threshhold x^c. Very few values are below x^c and will be changed by the process. On the integers, the critical value is very near, but not equal to 2/3.[6] However, in spite of hundreds of papers on BS, even such basic statements have not yet been proved with mathematical rigor, cf. ref. 13.

The Discrete Bak–Sneppen model. Since BS is hard to analyze, a simplified model DBS with only two states 0 and 1 was suggested by Barbay and Kenyon.[4] These states represent fitness in $[0, p[$ and in $[p, 1]$, respectively, where p is taken as parameter. When we compare with CP, species with small fitness correspond to sick individuals – they are "infectious" and trigger the process. So *let us turn the scale, taking state 1 for fitness in* $[0, p[$ *and state 0 for fitness in* $[p, 1]$. This notation is chosen so that the all-zero configuration is not changed by the process, which is a natural property for such models.

Assume we have a finite graph, and take an infection parameter p in [0,1] as in CP. In each time step, we choose randomly one species. If it is

in state 0, nothing happens. If it is in state 1, then the states of this species and all its neighbors are renewed. Each of these vertices is independently assigned value 1 with probability p, and value 0 with probability $1-p$. This process acts similarly as BS, but instead of organizing its critical state, the 1's will die out for small p, and they should spread around for large p, as in CP. In-between there should be a critical value p^c.

Meester and Znamenski[12] verified this for the process on a finite circle, i.e. N cyclically arranged sites. When all states were 0, they continued the process with a random site of state 0 and its neighbors. They proved that if p is close enough to 1, then the long-term average fitness of every site is greater ε where $\varepsilon > 0$ depends on p, but not on the site and not on N.

Main result of the paper. The aim of this note is to show, with a rather simple argument, that the discrete Bak–Sneppen model behaves exactly like the contact process, on an arbitrary graph. Thus all results which have been shown for CP will immediately extend to DBS, and it is not necessary to develop special theory for the new model.

However, it is not that easy. Since CP and BS come from different scientific communities, their settings are different. We can only compare CP with DBS when both have the same setting. We shall consider four versions, and each time we shall show that CP is equivalent to DBS. In the difficult case of interacting particle systems we can apply general criteria developed by Sudbury and Lloyd[15] and Bandt.[3] For the cellular automata version, we give definitions of CP and DBS which seem to be new, and a very simple argument. A message of this paper is that the equivalence of processes, like CP and DBS, is stronger than the technical details of different settings (discrete or continuous time, local or parallel action, random neighbor or all neighbors).

What does "equivalence" mean? Let us fix a finite graph $G=(V, E)$ with $n=|V|$ vertices. We have 2^n possible 0-1-configurations which we denote by letters η, ψ. A configuration η is often identified with the corresponding set of vertices with state 1, so η is just a subset of V.[9,10] The random processes which we consider in discrete time are Markov chains, given by a transition matrix $(p_{\eta\psi})$ with 2^n rows and columns. Let $C=C_p$ and $D=D_p$ denote the transition matrices for CP and DBS, respectively. We shall prove that these matrices are conjugate, where the conjugacy is given by another transition matrix $T=T_p$ called *thinning operator.* A similar construction can be done when the graph is infinite and the matrices are replaced by operators. Here is our main result.

Theorem. Let C and D be the Markov operators which correspond to CP and DBS with the same parameter p, respectively. Then

$$D \cdot T = T \cdot C.$$

This is a general statement on conjugacy of certain matrices. A similar connection is known between coalescing and annihilating random walk.[1] The theorem holds on arbitrary graphs, and will be proved for four different versions of the processes. Let us first define the thinning operator. $T = T_p$ acts on each vertex independently. If the vertex is in state 0, nothing happens. If the vertex is in state 1, it will remain so with probability p, and will go to state 0 with probability $q = 1 - p$. Thus if η is a subset of V with n sites, then $T_p(\eta)$ is a random subset of η. For each $\psi \subseteq \eta$ with k points we have $T_p(\eta) = \psi$ with probability $p^k q^{n-k}$. Thus the number of elements of $T_p(\eta)$ has binomial distribution with parameters n, p.

Thinnings and their connection with duality of processes were studied in a more general setting by Lloyd and Sudbury[15] and Bandt.[3] The theorem says that CP acts on any initial configuration in the same way as DBS acts on the thinned configuration. This is a very tight connection between the two processes. As a corollary we show that CP and DBS will either both die out for a parameter p, or will both survive with positive probability. Thus the critical parameter p^c will be the same for both processes, for an arbitrary graph.

Actually, this was the starting point of the present paper. With Tyll Krüger from Bielefeld we studied CP and DBS already in 1999 by computer simulation and found that the critical parameters are the same. I would like to thank Tyll for introducing me to the subject of self-organized criticality.

Cellular automata. First we shall study a cellular automata version of the processes which apparently has never been in the focus of probabilists working with CP and physicists working with BS. We take a locally finite graph G (possibly with infinitely many vertices), and in each discrete time step we perform parallel processing of all 1's and all their neighbors. When DBS is considered as a discretization of BS, this version is appealing since the "minimum site" is certainly processed at each step. Moreover, it is very clear how to define DBS as a cellular automaton: each site with state 1 and each neighbor of such a site should be renewed. Each of these vertices is independently assigned 1 with probability p and 0 with probability $q = 1 - p$. More formally, for a configuration ψ of 1's consider the neighborhood

$$U(\psi) = \psi \cup \{u \mid \text{there is } v \in \psi \text{ with } (v, u) \in E\}.$$

The new configuration $D(\psi)$ is the p-thinning of $U(\psi)$, that is $D(\psi) = T(U(\psi))$.

Let us define the operator D more formally. If $X = \{0, 1\}^V$ denotes the space of all configurations ψ, consider the linear space $\mathbb{R}^X$ of all linear combinations $\mu = \sum_{\psi \in X} c_\psi \cdot \psi$ with real coefficients c_ψ. Our Markov operators will be linear operators on this vector space. They act in particular on the simplex S of all probability distributions on X. The set $S \subset \mathbb{R}^X$ contains those linear combinations μ for which $c_\psi \geqslant 0$ and $\sum c_\psi = 1$. When we identify η with $1 \cdot \eta \in S$, we consider X as a subset of the simplex S.

The mappings T and U are defined on X, with values in S. They extend to form linear operators on $\mathbb{R}^X$ in the natural way: $T(\sum c_\psi \psi) = \sum c_\psi T(\psi)$. Clearly, T and U become Markov operators, i.e. they map S into S. According to the above discussion, the Markov operator D of DBS is given by

$$D = T \cdot U. \tag{1}$$

To define CP as a cellular automaton, we note that a site with k occupied neighbors is subject to k attacks of infection. If we postulate that "infection prevents recovery", then we have in the case of k occupied neighbors

$$1 \to 0 \text{ for an occupied site with probability } q^{k+1},$$
$$0 \to 1 \text{ for an unoccupied site with probability } 1 - q^k.$$

This cellular automata rule of CP is realized for a configuration η as follows. For each site v with $\eta(v) = 1$ independently we take a random number $x(v)$ where $x(v) = 1$ (the site remains infectious) with probability p, and $x(v) = 0$ (the site tends to recover) with probability q. For an arbitrary site w, let $C(\eta)(w) = 1$ if either $\eta(w) = 1 = x(w)$, or if there is an edge $(v, w) \in E$ with $\eta(v) = 1 = x(v)$. If there is no such edge, and $x(w) = 0$, let $C(\eta)(w) = 0$. All other sites remain unchanged.

Let us define the Markov operator C of CP formally. The set ψ of all $v \in \eta$ with $x(v) = 1$ is a p-thinning of η, that is $\psi = T(\eta)$. The set $C(\eta)$ is the neighborhood of ψ:

$$C(\eta) = U(\psi) = U(T(\eta)) \text{ or short: } C = U \cdot T. \tag{2}$$

Now our theorem follows directly from (1) and (2):

$$D \cdot T = T \cdot U \cdot T = T \cdot C. \tag{3}$$

Using standard methods,[9] the proof can be extended to countable locally finite graphs, like $\mathbb{Z}^d$, the standard graph for cellular automata.

Remark. $DT = TC$ implies $D^n T = TC^n$ by induction. Let us take an arbitrary initial configuration η, and consider the thinning of the random configuration $C^n(\eta)$ obtained by running the contact process n time steps. We proved that this coincides with the random configuration $D^n(\xi)$, where $\xi = T(\eta)$ is the thinning of η. Here η and ξ denote random configurations, more precisely, *distributions of configurations*. Relations between Markov operators imply relations between distributions.

If we realize both processes CP and DBS on a common probability space Ω, using one random number for each site and each time step, the processes DT and TC will *not result in identical configurations* $\psi(\omega)$ *for almost all* $\omega \in \Omega$. In fact, ω and the random element $\overline{\omega}$ for the thinning operator are chosen independently. In (3), $D \cdot T$ equals $T_\omega U T_{\overline{\omega}}$ while $T \cdot C$ is $T_{\overline{\omega}} U T_\omega$.

Example. Consider a graph with two vertices u, v connected by an edge, and the initial configuration $\eta(u) = 0, \eta(v) = 1$. The thinning operator is realized by two random numbers $t(u), t(v)$ which are 1 (keep the point) with probability p and 0 (cancel the point) with probability $q = 1 - p$. Both DBS and CP are realized by two other random numbers $x(u), x(v)$ with the same distribution. In this case 1 means low fitness or infection, and 0 means high fitness or recovery. All four random numbers are chosen independently.

Now the configurations $\xi = T(\eta), \psi = DT(\eta)$ and $\phi = TC(\eta)$ are *random configurations* since they depend on random numbers. It is easy to see that $\psi(v) = \phi(v)$ since this will be 1 if and only if $t(v) = x(v) = 1$. However, $\psi(u) = 1$ means $t(v) = x(u) = 1$ and $\phi(u) = 1$ means $t(u) = x(v) = 1$. Thus the events $\psi(u) = 1$ and $\phi(u) = 1$ have the same probability p^2 but they are different and even independent since they depend on different random numbers. This example holds with slight modification for all other versions below.

Discrete time, all neighbors. Now we consider the original version of DBS[4,12] on a finite directed graph $G = (V, E)$. In each step, one vertex v of V is chosen, where each vertex u has probability p_u, and $\sum_{u \in V} p_u = 1$. In the uniform case, $p_u = 1/n$. We then consider the neighborhood $U(v)$ consisting of v and all w for which there is an edge (v, w). If $\eta(v) = 1$ then for DBS, each point of $U(v)$ is assigned independently the state 1 with probability p and state 0 with probability q. For CP we assign 1 to all elements of $U(v)$ with probability p, and we assign 0 to v with probability q. The states of the other vertices remain unchanged. If $\eta(v) = 0$ then both processes will change nothing.

The Markov operators for this version will have the form $D = \sum_{v \in V} p_v D_v$ and $C = \sum_{v \in V} p_v C_v$ where C_v, D_v are Markov operators which change only the state of the vertices in $U(v)$. In order to prove our theorem, it is enough to show $D_v \cdot T(\eta) = T \cdot C_v(\eta)$ for every $v \in V$ and every 0-1-configuration η on V. So let us fix $v \in V$ and assume $\eta(v) = 1$ because otherwise the equation needs no proof. Moreover, since both operators $D_v T$ and TC_v result in a p-thinning of η on $V \backslash U(v)$, we consider only the action on $U(v)$.

Consider $D_v T$. With probability p, the first operator T will leave $\eta(v) = 1$ unchanged, and D_v will then yield a p-thinning of the all-one-configuration on $U(v)$. With probability q, the operator T will cancel v from η, and D_v will not change anything. In this case, $D_v T(\eta)(v) = 0$, and $D_v T(\eta)$ is a p-thinning of η on $V \setminus \{v\}$.

Now consider TC_v. With probability q, the operator C_v will just provide recovery to the point v, so that $TC_v(\eta)(v) = 0$ and $TC_v(\eta)$ is a p-thinning of η on $V \setminus \{v\}$. With probability p, the operator C_v will bring infection from v to $U(v)$, and TC_v results in a p-thinning of the all-one-configuration on $U(v)$. The theorem is proved for the second version.

Discrete time, one random neighbor. Now we consider both processes in discrete time and with the original setting of CP where infection involves only one random neighbor. For the Bak–Sneppen model with continuous fitness, one-neighbor versions were studied for the mean-field case[5] and on the line.[8,11] This model can be formulated with the random choice of an edge $e = (v, w)$ from the finite directed graph, where v is the infectious site and w the neighbor to be infected. Here e is chosen with probability p_e, and $\sum_{e \in E} p_e = 1$.

In this setting, $C = \sum p_e C_e$, $D = \sum p_e D_e$ where C_e, D_e change only the states at v and w. To show the theorem, we verify $D_e T(\eta) = TC_e(\eta)$ for each edge e and each configuration η. The assertion holds for $\eta(v) = 0$ since both C_e and D_e will have no effect. For $\eta(v) = 1$ the proof proceeds as above.

There is a generalization of both the all-neighbor and one-neighbor setting. One can take families of neighborhoods $N(v)$ for each site v. In a time step, one point v and a corresponding neighborhood N is chosen with probability $p_{v,N}$. CP and DBS can be defined similarly as above, and the theorem can be proved, too.

Continuous time, one random neighbor. This is the original mathematical version of CP as an interacting particle system, see refs. 9, 10. Since it works for infinite locally finite graphs as $\mathbb{Z}^d$, this seems the most reasonable setting from the mathematical point of view, but it requires semigroups $S^t, t > 0$ of appropriate operators on Banach spaces. Fortunately, our theorem in this case is a special case of theorem 1.4 in ref. 3. There

we studied processes which change the states 0,1 on any two neighboring vertices of an undirected graph by the rules $11 \to_a 00$, $11 \to_c 01, 10 \to_d 00$, $10 \to_e 01$, and $10 \to_g 11$. Here $a, \ldots, g$ denote the rates for the corresponding change. For CP we have $a = e = 0, c = d = q$ and $g = p$. For DBS we have $a' = d' = q^2, c' = 2pq$ and $e' = pq, g' = p^2$. In both cases, the sum of rates is $1 + q$. Consider $x = d - 2a - c - g = -p$ and $x' = d' - 2a' - c' - g' = -1$. Then $x' = x/p$ and $g' = g \cdot p$ which by theorem 1.4 of ref. 3 implies the above theorem. This was done only for undirected graphs, but can be extended to the directed case.

Corollary. Starting from a configuration with a single 1, the probabilities that CP and DBS died out after n steps differ at most by the factor p.

Proof. Let $\eta(v) = 1$ and $\eta(u) = 0$ for all $u \neq v$. Our theorem implies $D^n T = T C^n$ for all n. Thus, using the independence of thinning and CP resp. DBS, and $p = P\{T(\eta) = \eta\}$, we get two equations for the survival probabilities:

$$P\{TC^n(\eta) \neq \mathbf{0}\} = P\{D^n T(\eta) \neq \mathbf{0}\} = pP\{D^n(\eta) \neq \mathbf{0}\},$$

$$P\{C^n(\eta) \neq \mathbf{0}\} \geqslant P\{TC^n(\eta) \neq \mathbf{0}\} \geqslant pP\{C^n(\eta) \neq \mathbf{0}\}.$$

This completes the proof. For interacting particle systems, a formula connecting the number of remaining 1's for C^t and D^t at any time t was given in Chapter 6 of ref. 3. ∎

Our note indicates that duality and thinning of interacting particle systems tend to carry over to discrete time and cellular automata versions of the processes under study.

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